PMM U.S.S.R.,Vol.45,pp.445-448
Copyright Pergamon Press Ltd.1982.Printed in U.K.
0021-8928/82/4 0445 $7.50/0

UDC 531.38

DOMAIN OF FEASIBLE MOTIONS IN CERTAIN MECHANICAL SYSTEMS*

T.I. POGOSIAN and M.P. KHARLAMOV

Domains of feasible motions - projections of integral manifolds onto a configuration space - are studied. It is shown that in systems with symmetry and in Liouville systems the reorganization of such domains is possible only simultaneously with the reorganization of the integral manifolds. An example is presented of a system with gyroscopic forces, in which the domains of feasible motions change topological type outside a bifurcation set.

1. Let $q_1, \ldots, q_n$ be generalized coordinates and $q_1^{\cdot}, \ldots, q_n^{\cdot}$ be generalized velocities of a mechanical system defined by the Lagrange function

$$L(\mathbf{q}, \mathbf{q}^{\cdot}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\mathbf{q}) q_i^{\cdot} q_j^{\cdot} + \sum_{i=1}^{n} b_i(\mathbf{q}) q_i^{\cdot} + c(\mathbf{q}) \tag{1.1}$$

A curve $(q_1(t), \ldots, q_n(t))$ satisfying the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}}\left(\mathbf{q}(t), \frac{d\mathbf{q}(t)}{dt}\right) - \frac{\partial L}{\partial q_i}\left(\mathbf{q}(t), \frac{d\mathbf{q}(t)}{dt}\right) = 0 \tag{1.2}$$

is called a motion in the system being studied, while the corresponding curve $(q_1(t), \ldots, q_n(t), dq_1(t)/dt, \ldots, dq_n(t)/dt)$ in the phase space $(\mathbf{q}, \mathbf{q}^{\cdot})$ is called a phase trajectory. Let Eq.(1.2) have the first integrals

$$K_\alpha(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot}) = k_\alpha \ (\alpha = 1, \ldots, m) \tag{1.3}$$

The subset cut out in the phase space by relations (1.3) is called an integral manifold. We denote it $I_{k_1,\ldots,k_m}$. An integral manifold consists wholly of phase trajectories.

The study of a system's phase trajectories is connected closely with the classification of the integral manifolds. In its own turn the latter is based on the study of a bifurcation set Σ, viz., a set of points in the space of constants of integrals (1.3), upon passing through which the topological type of the integral manifolds is changed /1/. If set Σ has been found and if the topological or differentiable type of the corresponding integral manifold has been indicated for each connected component $\mathbf{R}^m \setminus \Sigma$, then we say that the problem's phase topology has been investigated. However, knowledge of the types of the phase trajectories does not always yield an idea of the real behavior of the system under examination. In connection with this it becomes important to study how the motions, i.e., the projections of the phase trajectories onto the configuration space, are constructed. It is evident that for prescribed $k_1, \ldots, k_m$ the corresponding motions pass through those and only those points $(q_1, \ldots, q_n)$ at which equality (1.3) is solvable relative to $q_1^{\cdot}, \ldots, q_n^{\cdot}$. We denote the set of such points by $M_{k_1,\ldots,k_m}$ and we call it the domain of feasible motions (DFM). It is clear that the DFM is a projection of integral manifold $I_{k_1,\ldots,k_m}$ onto the configuration space. For a specified point of the DFM the vector $(q_1^{\cdot}, \ldots, q_n^{\cdot})$, namely, a solution of (1.3), is called an admissible velocity. The points of the DFM at which the topological type of the set of admissible velocities is changed (the critical values of the projection mentioned) form the generalized boundary of the DFM. The generalized boundary contains within itself the topological boundary. We pose the problem of classifying the types of DFM with due regard to the generalized boundaries.

2. Let us briefly outline the situation obtaining in systems with symmetry (see /2/ for the details). We consider a natural mechanical system with a configuration space M. It is

*Prikl.Matem.Mekhan.,45,No.4,605-610,1981 specified by a Riemann metric on M (by the scalar product $\langle , \rangle_x$ in the tangent spaces to M, depending smoothly on point $x \in M$) and by a function V on M. The Lagrange function of such a system $L(v) = {}^1/_2 \langle v, v\rangle_x - V(x)$ (v is a vector tangent to M at point x) has the form (1.1) in local coordinates, and all $b_i(\mathbf{q}) \equiv 0$. Let a one-parameter diffeomorphism group $\{g^\tau\}$ act freely on M, preserving the function V and the scalar product. Then in addition to the energy integral

$$H(v) \equiv {}^1/_2 \langle v, v\rangle_x + V(x) = h \tag{2.1}$$

the system admits of an integral

$$J(v) \equiv \langle u, v\rangle = s \tag{2.2}$$

linear in the velocities (h and s are constants, $u(x) = (d/d\tau)_{\tau=0} g^\tau(x)$ is a nonzero vector field generating the group's action). For fixed $x \in M$, $s \in \mathbf{R}$ the set $J^{-1}(s)$ is a hyperplane in the space of tangent vectors to manifold M at point x. We define a vector field $u^s(x)$ such that $u^s(x) \in J^{-1}(s)$ and $u^s(x) \perp J^{-1}(0)$ in the Riemann metric prescribed. The function $V_s(x) = V(x) + {}^1/_2 \langle u^s, u^s\rangle_x$ is called the reduced potential.

In this problem the DFM have the form $M_{h,s} = \{x \in M : V_s(x) \leqslant h\}$. Obviously, a change in the type of $M_{h,s}$ is possible only when h passes through the critical value V_s. The latter signifies, however, that (h, s) intersects the bifurcation set of integrals (2.1), (2.2) /1/. Thus, reorganizations of the DFM are possible only on the bifurcation set.

3. Let us consider a system with two degrees of freedom, of Liouville type. In suitable coordinates (x, y) the Lagrange function of such a system is

$$L = \frac{1}{2}[U(x) + V(y)][\alpha^2(x)x^{\cdot 2} + \beta^2(y)y^{\cdot 2}] - \frac{\varphi(x) + \psi(y)}{U(x) + V(y)}$$

Once more this is a function of form (1.1) with $b_i(\mathbf{q}) \equiv 0$. The system admits of two quadratic first integrals

$$\begin{aligned} H &\equiv \frac{1}{2}(U + V)(\alpha^2 x^{\cdot 2} + \beta^2 y^{\cdot 2}) + \frac{\varphi(x) + \psi(x)}{U + V} = h \\ K &\equiv \frac{1}{2}(U + V)(V\alpha^2 x^{\cdot 2} - U\beta^2 y^{\cdot 2}) + \frac{\varphi V - \psi U}{U + V} = k \end{aligned} \tag{3.1}$$

From these we find

$$ {}^1/_2 (U + V)^2\alpha^2 x^{\cdot 2} = hU + k - \varphi, \quad {}^1/_2 (U + V)^2\beta^2 y^{\cdot 2} = hV - k - \psi$$

so that the corresponding DFM is determined by the inequalities

$$M_{h,k} = \{(x, y) : hU(x) + k - \varphi(x) \geqslant 0,\ hV(y) - k - \psi(y) \geqslant 0\}$$

It is obvious that a reorganization of $M_{h,k}$ is possible only when there is a critical point of one of the functions $hU + k - \varphi$ or $hV - k - \psi$ on the boundary of $M_{h,k}$. Let us show that the point (h, k) belongs to the bifurcation set of integrals (3.1). For example, let the point (x, y) be such that

$$hU(x) + k - \varphi(x) = 0, \quad hV(y) - k - \psi(y) = 0, \quad hV'(y) - \psi'(y) = 0 \tag{3.2}$$

We set $x^\cdot = y^\cdot = 0$. By virtue of (3.2)

$$H(x, y, 0, 0) = h, \quad K(x, y, 0, 0) = k \tag{3.3}$$

In addition, the partial derivatives of functions H and K at such a point are

$$\begin{aligned} &H_{x^\cdot} = H_{y^\cdot} = 0, \quad K_{x^\cdot} = K_{y^\cdot} = 0, \quad H_y = K_y = 0 \\ &H_x = -\frac{1}{U + V}(hU' - \varphi'), \quad K_x = -\frac{V}{U + V}(hU' - \varphi') \end{aligned}$$

In particular

$$\operatorname{grad} H - V(y)\operatorname{grad} K\,|_{(x, y, 0, 0)} = 0$$

Thus, at the point being examined the integrals (3.1) are dependent, but then, on the strength of (3.3), $(h, k) \in \Sigma$. Consequently, here too reorganizations of the DFM can obtain only on the bifurcation set. An elementary case has been considered above. Liouville systems of arbitrary dimension have been studied in /3/. Concrete examples of two-dimensional systems have been investigated in detail in /3,4/.

4. Let us dwell on the problem of the motion of a asymmetric rigid body, fixed at the center of mass, in a Newtonian force field. The results extend the investigations in /5/. All notation is consistent with the latter paper. A lowering of the order with the aid of an area integral (whose constant is henceforth denoted by s) leads to a mechanical system on the Poisson sphere

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{4.1}$$

(ν_i are components in the moving trihedron of the direction vector of "the center of mass-center of attraction" axis). When $s \neq 0$ the system is not natural, i.e., the Lagrangian (the Routh function) of form (1.1) in the local coordinates on sphere (4.1) contains in essential manner summands linear in the generalized velocities. By $a > b > c$ we denote quantities inverse to the principal central energy moments of the body and on sphere (4.1) we introduce the local coordinates λ, μ

$$\nu_1^2 = \frac{(a-\lambda)(a-\mu)}{(a-b)(a-c)}, \quad \nu_2^2 = \frac{(\lambda-b)(b-\mu)}{(a-b)(b-c)}, \quad \nu_3^2 = \frac{(\lambda-c)(\mu-c)}{(a-c)(b-c)}$$

The first integrals of the reduced system take the form /5/

$$H \equiv \frac{\lambda-\mu}{8\lambda\mu}\left(\frac{\lambda\lambda^{\cdot 2}}{f(\lambda)} - \frac{\mu\mu^{\cdot 2}}{f(\mu)}\right) + \frac{abcs^2}{2\lambda\mu} + \frac{\lambda\mu}{2} = h \tag{4.2}$$

$$K \equiv \frac{\lambda-\mu}{8\lambda^2\mu^2}\left(\frac{\lambda^2\lambda^{\cdot 2}}{f(\lambda)} - \frac{\mu^2\mu^{\cdot 2}}{f(\mu)}\right) - \frac{s}{2\lambda^2\mu^2}\left(\lambda^2\sqrt{-\frac{f(\mu)}{f(\lambda)}}\lambda^{\cdot} - \right.$$
$$\left.\mu^2\sqrt{-\frac{f(\lambda)}{f(\mu)}}\mu^{\cdot}\right) + \frac{s^2}{2\lambda^2\mu^2}[(bc+ca+ab)\lambda\mu - abc(\lambda+\mu)] +$$
$$\frac{1}{2}[\lambda+\mu-(a+b+c)] = k, \quad f(\tau) = (a-\tau)(b-\tau)(c-\tau)$$

Their bifurcation set is shown on Fig.1 for small values of s. In regions 1° and 2° the integral manifolds $I_{h,k}$ consist of two nonintersecting two-dimensional tori not supporting conditionally-periodic motions, while in region 3° it consists of four such tori. In region 4°, $I_{h,k} = \varnothing$ /5/. In this problem let us investigate the DFM, i.e., the domains on sphere (4.1), swept by a unit vector fixed in space.

We rewrite (4.2) as

$$u^2 + v^2 = (2\lambda\mu h - \lambda^2\mu^2 - abcs^2) / (\lambda-\mu) \tag{4.3}$$

$$\frac{1}{\lambda\mu^2}\left[u - \frac{s\sqrt{-\lambda f(\mu)}}{\lambda-\mu}\right]^2 + \frac{1}{\lambda^2\mu}\left[v + \frac{s\sqrt{\mu f(\lambda)}}{\lambda-\mu}\right]^2 = \frac{2k+a+b+c-s^2-\lambda-\mu}{\lambda-\mu} \tag{4.4}$$

$$u = \lambda^{\cdot}\sqrt{\lambda/f(\lambda)}/2, \quad v = \mu^{\cdot}\sqrt{-\mu/f(\mu)}/2$$

Thus, the set of admissible velocities at point (λ, μ) is the set of points of intersection of circle (4.3) and ellipse (4.4). Obviously, point (λ, μ) belongs to the generalized boundary of a DFM if and only if the corresponding curves (4.3) and (4.4) have at least one point of tangency. Writing the proportionality condition for the gradients with respect to u and v, we find that at the point of tangency

$$u = \frac{s\tau\sqrt{-\lambda f(\mu)}}{(\lambda-\mu)(\tau-\mu)}, \quad v = \frac{s\tau\sqrt{\mu f(\lambda)}}{(\lambda-\mu)(\lambda-\tau)} \tag{4.5}$$

(τ is the gradients proportionality coefficient). We denote

$$x = (\lambda-\tau)(\tau-\mu), \quad y = (\lambda-\tau)-(\tau-\mu) \tag{4.6}$$

The substitution of (4.5) into (4.3) and (4.4) leads to the equations

$$x^2 + [2h - (2k+a+b+c)\tau + \tau^2]x + s^2 f(\tau) = 0, \qquad y = x\frac{(2k+a+b+c-2\tau)x - s^2 f'(\tau)}{x^2 - s^2 f(\tau)} \tag{4.7}$$

from which the functions $x(\tau)$, $y(\tau)$ are determined. Finally, from (4.6) we find the equation of the generalized boundary

$$\begin{Bmatrix}\lambda \\ \mu\end{Bmatrix} = \tau + \frac{1}{2}[y(\tau) \pm \sqrt{y^2(\tau) + 4x(\tau)}] \tag{4.8}$$

We note that system (4.3), (4.4) is completely analytically solvable at the points of the DFM's generalized boundary and we can show that solution (4.5) is unique if $x < 0$. Under the condition $x > 0$ the ellipse and the circle have two more points of intersection. We present the investigation's result without dwelling on the quite cumbersome analysis of Eqs.(4.7) and (4.8) (in particular, on the study of the singular case $x = 0$ when the appearance of extraneous solutions is possible).

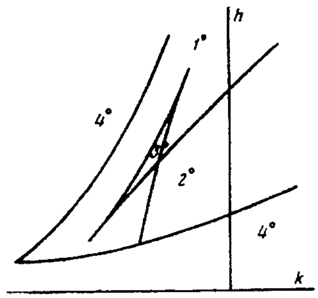

Fig.1

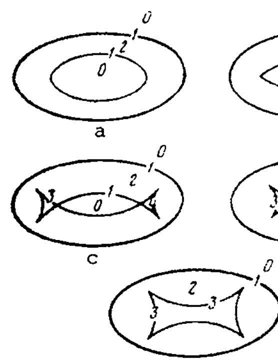

Fig.2

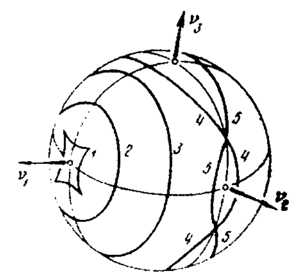

Fig.3

Fig.2 shows five basic types of DFM arising on sphere (4.1). The numbers denote the number of admissible velocities at the corresponding points of the DFM or of its generalized boundary. At the cusps and the selfintersection points of the generalized boundary we have two admissible velocties. It is easy to understand how these cases are obtained under the projection of the integral torus onto sphere (4.1) and to see how the images of the conditionally-periodic trajectories on the torus are constructed. In regions 1° and 2° (see Fig.1) the DFM consists of two components of the kind indicated, while in region 3°, of four such components. In this connection, their partial or complete mutual overlappings are possible. An exhaustive description of all situations is not possible within the scope of the present paper. We merely present an example of one of the most complex situations relating to region 3° (Fig.3). Curves 1 and 4 bound a domain of type 2,e; curves 2 and 3, a domain of type 2,a. The DFM is symmetric relative to a section of the sphere by the plane $\nu_1 = 0$ (consequently, curve 5 is analogous to curve 4).

We stress that all the situations shown in Fig.2 result form a prescribed area constant in each of regions $1^\circ - 3^\circ$ into which the bifurcation set divides the plane of constants of the integrals. Thus, in the problem at hand the DFM undergo reorganizations at the places where the corresponding integral manifolds do not change their differentiable type.

REFERENCES

1. SMALE S., Topology and Mechanics. I, II. Invent. Math., I: Vol.10, 1970; II: Vol.10, 1970.
2. TAMARINOV Ia.V., On the investigation of the phase topology of compact configurations with symmetry. Vestn. Moskovsk. Gos. Univ., Ser. Mat., Mekh., No.5, 1973.
3. ALEKSEEV V.M., Generalized spatial problem of two fixed centers. Classification of motions. Biull. Inst. Teor. Astorn. Akad. Nauk SSSR, Vol.10, No.4, 1965.
4. KHARLAMOV M.P., Phase topology of one integrable case of motion of a rigid body. In: Mechanics of a Rigid Body, No.11. Kiev, NAUKOVA DUMKA, 1979.
5. POGOSIAN T.I. and KHARLAMOV M.P., Bifurcation set and integral manifolds of the problem concerning the motion of a rigid body in a linear force field. PMM Vol.43, No.3, 1979.

Translated by N.H.C.